

# MONGE-AMPÈRE TYPE EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS ON RIEMANNIAN MANIFOLDS

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**ABSTRACT.** In this paper, we consider the global regularity for Monge-Ampère type equations with the Neumann boundary conditions on Riemannian manifolds. It is known that the classical solvability of the Neumann boundary value problem is obtained under some necessary assumptions. Our main result extends the main theorem from the case of Euclidean space  $R^n$  in [11] to Riemannian manifolds.

## 1. INTRODUCTION

The main purpose of this paper is to study the Neumann boundary value problem for the Monge-Ampère type equations on Riemannian manifolds. Let  $(M^n, g)$  be an  $n \geq 2$  dimensional smooth Riemannian manifold.  $S_2M^n$  is the bundle of symmetric  $(0,2)$  tensor on  $M^n$ , and  $\Omega \subset M^n$  is a compact domain with smooth boundary  $\partial\Omega$ . We consider the equation

$$(1.1) \quad \det[(\nabla^2 u - A(x, u, \nabla u))g^{-1}] = B(x, u, \nabla u) \quad \text{in } \Omega,$$

together with the Neumann boundary condition

$$(1.2) \quad \nabla_\nu u = \varphi(x, u) \quad \text{on } \partial\Omega,$$

where  $A : \bar{\Omega} \times \mathbb{R} \times T_x M^n \rightarrow S_2M^n$ ,  $B \geq 0$  is  $C^\infty$  with respect to  $(x, z, p) \in \bar{\Omega} \times \mathbb{R} \times T_x M^n$ . Here  $T_x M^n$  denotes the tangent space at  $x \in M^n$ , and  $\nu$  is the unit inner normal vector field on  $\partial\Omega$ . As customary  $\nabla u$  and  $\nabla^2 u$  denote respectively the gradient vector and Hessian matrix of second derivatives of  $u$ . A solution  $u$  is elliptic, (degenerate elliptic), if the matrix  $\nabla^2 u - A(x, u, \nabla u)$  is positive, (non negative) definite.

There has been considerable research activity in recent years devoted to fully nonlinear elliptic, second order partial differential equations of the form (1.1), which arise in applications, notably in optimal transportation [22, 30] and also in reflector and refractor shape design problems [7, 33, 34].

The Yamabe problem on manifold with boundary was studied by Escobar [2], he showed that almost every compact Riemannian manifold was equivalent to the constant scalar curvature manifold, whose boundary was minimal. In fact the problem can be reduced to solve the semilinear elliptic critical Sobolev exponent equation with Neumann boundary condition. The Neumann boundary problem of linear and quasilinear elliptic equation was widely studied for a long time, readers can see the recent book written by Lieberman[16]. The motivation of studying the Neumann boundary value problem for Monge-Ampère type equations comes from its application in conformal geometry. Such a prescribed mean curvature problem in conformal geometry was first proposed in [14].

There are also many known results about the fully nonlinear elliptic equations on Riemannian manifolds. For example, Guan and Li in [5] extended the well-known result for Monge-Ampère equation with Dirichlet boundary condition in  $\mathbb{R}^n$ . For more results, we refer readers to the articles [1, 4, 6] and references therein.

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*Date:* November 1, 2016.

*Key words and phrases.* second derivative estimate, Monge-Ampère type equation, Neumann problem, Riemannian manifold.

In this paper, we shall derive a priori second-order estimates for solutions of the Neumann boundary value problem (1.1)-(1.2) on Riemannian manifolds. It is well known that these estimates yield regularity and existence results. For this aim, the regularity of solutions depends on the behaviour of the matrix  $A$  with respect to the  $p$  variables. We call the matrix  $A$  regular if  $A$  is co-dimension one convex with respect to  $p$ , in the sense that

$$(1.3) \quad \nabla_{p_k p_l} A_{ij}(x, z, p) \xi_i \xi_j \eta_k \eta_l \geq 0,$$

for all  $(x, z, p) \in \Omega \times \mathbb{R} \times T_x M^n$ ,  $\xi, \eta \in T_x M^n$ ,  $\xi \perp \eta$ . If (1.3) is replaced by

$$(1.4) \quad \nabla_{p_k p_l} A_{ij}(x, z, p) \xi_i \xi_j \eta_k \eta_l \geq c_0 |\xi|^2 |\eta|^2,$$

for some  $c_0 > 0$ , then the matrix  $A$  is called strictly regular. Conditions (1.3) and (1.4) were introduced in [22, 27] and called A3w, A3 respectively. Loeper in [21] showed that the condition A3w was indeed the necessary and sufficient condition for regularity. One can not expect regularity without this condition. A case of special interest for applications is the generalization of Brenier's cost to Riemannian manifolds. Existence and uniqueness of optimal maps in that case was established by McCann [23].

As with [10, 29], we also need monotone assumptions about  $A$ ,  $B$  and  $\varphi$ . The matrix  $A$  is non-decreasing (strictly increasing) with respect to  $z$ , if

$$(1.5) \quad \nabla_z A_{ij}(x, z, p) \xi_i \xi_j \geq 0 (> 0),$$

for all  $(x, z, p) \in \Omega \times \mathbb{R} \times T_x M^n$ ,  $\xi \in \mathbb{R}^n$ . The inhomogeneous term  $B$  is non-decreasing (strictly increasing) with respect to  $z$ , if

$$(1.6) \quad \nabla_z B(x, z, p) \geq 0 (> 0),$$

for all  $(x, z, p) \in \Omega \times \mathbb{R} \times T_x M^n$ . The function  $\varphi$  defined on the boundary is called non-decreasing (strictly increasing) with respect to  $z$ , if

$$(1.7) \quad \nabla_z \varphi(x, z) \geq 0 (> 0),$$

for all  $(x, z) \in \partial\Omega \times \mathbb{R}$ .

As well, we need to assume a kind of global barrier condition called the uniformly  $A$ -convexity in [27] for the domain  $\Omega$ , namely that there exists a defining function  $\phi \in C^2(\bar{\Omega})$ , satisfying  $\phi = 0$  on  $\partial\Omega$ ,  $\nabla\phi \neq 0$  on  $\partial\Omega$  and  $\phi < 0$  in  $\Omega$ , together with the inequality

$$(1.8) \quad \nabla_{ij} \phi - \nabla_{p_k} A_{ij}(x, u, \nabla u) \nabla_k \phi \geq \delta_0 I,$$

in a neighbourhood  $\mathcal{N}$  of  $\partial\Omega$ , where  $\delta_0$  is a positive constant,  $I$  denotes the identity matrix. The inequality (1.8) is trivially satisfied in the standard Monge-Ampère case which can be easily seen by taking  $\phi(x) = |x|^2$  in  $R^n$ . For the Monge-Ampère equations on manifolds, (1.8) is a natural condition for existence of global smooth solutions, called existence of a geodesic convex function on  $\Omega$  by Hong in [9]. By virtue of the uniformly  $A$ -convexity of the domain  $\Omega$ , for example,

$$(1.9) \quad \phi = -ad + bd^2,$$

where  $a$  and  $b$  are positive constants and  $d(x) \triangleq \text{dist}(x, \partial\Omega)$  denotes the distance function for  $\Omega$ , see [3, 30] for reference.

In order to achieve the second order derivative estimate under the necessary natural condition (1.3), we need to assume the existence of a supersolution to the problem (1.1)-(1.2) satisfying

$$(1.10) \quad \det[(\nabla^2 \bar{u} - A(x, \bar{u}, \nabla \bar{u}))g^{-1}] \leq B(x, \bar{u}, \nabla \bar{u}) \quad \text{in } \Omega,$$

$$(1.11) \quad \nabla_\nu \bar{u} = \varphi(x, \bar{u}) \quad \text{on } \partial\Omega,$$

and a subsolution to the problem (1.1)-(1.2) satisfying

$$(1.12) \quad \det[(\nabla^2 \underline{u} - A(x, \underline{u}, \nabla \underline{u}))g^{-1}] \geq B(x, \underline{u}, \nabla \underline{u}) \quad \text{in } \Omega,$$

$$(1.13) \quad \nabla_\nu \underline{u} = \varphi(x, \underline{u}) \quad \text{on } \partial\Omega.$$

We now formulate the main results of this paper. First, the global second derivative estimate can be obtained as follows.

**Theorem 1.1.** *Assume that  $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$  is an elliptic solution of the problem (1.1)-(1.2) in a  $C^{3,1}$  uniformly  $A$ -convex domain  $\Omega \subset \mathbb{M}^n$ , where  $A \in C^2(\bar{\Omega} \times \mathbb{R} \times T_x M^n)$  is regular and non-decreasing,  $B > 0, \in C^2(\bar{\Omega} \times \mathbb{R} \times T_x M^n)$  and  $\varphi \in C^{2,1}(\partial\Omega \times \mathbb{R})$  are both non-decreasing. Suppose that there exists an elliptic supersolution  $\bar{u} \in C^2(\bar{\Omega})$  satisfying (1.10)-(1.11). Then we have the estimate*

$$(1.14) \quad \sup_{\Omega} |\nabla^2 u| \leq C,$$

where  $C$  is a constant depending on  $n, A, B, \Omega, \bar{u}, \varphi, \delta_0$  and  $|u|_{1;\Omega}$ .

Due to Theorem 1.1, we obtain the classical existence theorem for (1.1) and (1.2) under further hypotheses for the solution bounds and the gradient estimates. For the solution estimates, we can assume the existence of bounded subsolutions and supersolutions by virtue of the comparison principle. Under a further structural assumption on the matrix  $A$ ,

$$(1.15) \quad A(x, z, p) \geq -\mu_0[1 + |p|^2],$$

for all  $x \in \Omega, |z| \leq K, p \in T_x M^n$  and some positive constant  $\mu_0$  depending on the constant  $K$ , we can control the gradient of elliptic solution which has been proved in [8] and extends the results for the case of Euclidean space  $R^n$  in [13].

Combining the second derivative bounds with the lower order estimates, we can get the global second derivative Hölder estimates as in [17, 18, 19, 25] and establish the existence result by the method of continuity.

**Theorem 1.2.** *Under the assumptions of Theorem 1.1, suppose that either  $A, B$  or  $\varphi$  is strictly increasing. Assume that condition (1.15) holds and there is an elliptic subsolution satisfying (1.12) and (1.13). Then the Neumann boundary value problem (1.1)-(1.2) has a unique elliptic solution  $u \in C^{3,\alpha}(\bar{\Omega})$  for any  $\alpha < 1$ .*

The uniqueness of the solution follows from the comparison principle for the elliptic solution. The regularity for the solution  $u$  in Theorem 1.2 can be improved by the linear elliptic theory [3] if the data are sufficiently smooth. For example, if  $A, B, \varphi$  and  $\partial\Omega$  are  $C^\infty$ , then  $u \in C^\infty(\bar{\Omega})$ .

The paper is organized as follows. Section 2 devotes to some preliminary results, such as a comparison principle for the Neumann problem (1.1)-(1.2), the maximum modulus and the gradient estimate. In Section 3, we consider the second order derivative estimate in the interior of the domain and in a neighbourhood of the boundary successively. Then the proof of Theorem 1.1 is given, which is crucial for this paper. In Section 4, we provide the proof of Theorem 1.2.

## 2. PRELIMINARIES

In this section, first we recall some formulae for commuting covariant derivatives on  $M^n$ . Then we study the maximum modulus and gradient bounds for elliptic solutions of the Neumann boundary value problem (1.1)-(1.2). The maximum modulus is obtained from the assumed supersolution and subsolution by virtue of a comparison principle for the Neumann boundary value problem. The gradient bound has been established only using the ellipticity of the solution and a quadratic bound from below of the matrix  $A$ . In [8] we have obtained the gradient estimate for the degenerate elliptic solution of the problem (1.1)-(1.2), by using the ellipticity of the solution and a quadratic bound from below of the matrix  $A$ . Here we formulate the gradient estimate as a lemma without proof.

Let  $(M^n, g)$  be an  $n$ -dimensional Riemannian manifold. Throughout the paper,  $\nabla$  denotes the covariant differentiation on  $M^n$ . We choose a local orthonormal vector field  $\{e_1, \dots, e_n\}$  adapted to the Riemannian metric of  $(M^n, g)$  with its dual coframe  $\{\omega_1, \dots, \omega_n\}$ . Then we have

$$(2.1) \quad \nabla u = \nabla_j u \omega_j, \quad \nabla_{e_i} \nu = \nabla_i \nu_k e_k,$$

and

$$(2.2) \quad \nabla^2 u = \nabla_{ij} u \omega_i \omega_j,$$

where

$$(2.3) \quad \nabla_{ij} u = \nabla_i(\nabla_j u) - (\nabla_i e_j)u.$$

We recall that

$$(2.4) \quad \nabla_{ij} u = \nabla_{ji} u.$$

From the Ricci identity, we have

$$(2.5) \quad \nabla_{ijk} u - \nabla_{jik} u = R_{lkji} \nabla_l u,$$

where  $R_{ijkl}$  is the component of the Riemannian curvature tensor of  $(M^n, g)$ . The connection forms  $\{\omega_{ij}\}$  of  $(M^n, g)$  are characterized by the structure equations

$$\begin{aligned} d\omega_i &= - \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} &= - \sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l. \end{aligned}$$

We consider the distance function

$$(2.6) \quad d(x) = \text{dist}(x, x_0),$$

in a small ball  $B_r(x_0) = \{x \in \Omega, d(x) < r\}$ . By choosing  $r$  small enough we may assume  $d^2(x)$  is smooth and

$$(2.7) \quad \{\delta_{ij}\} \leq \{\nabla_{ij} d^2\} \leq 3\{\delta_{ij}\}$$

in  $B_r(x_0)$ .

From the equation (1.1), we have

$$(2.8) \quad F[u] = \ln \det[(\nabla^2 u - A(x, u, \nabla u))g^{-1}] = \tilde{B}(x, u, \nabla u),$$

where  $\tilde{B} = \ln B$ . We still denote

$$F^{ij} = \frac{\partial F}{\partial w_{ij}} = w^{ij}, \quad F^{ij,kl} = \frac{\partial^2 F}{\partial w_{ij} \partial w_{kl}} = -w^{ik} w^{jl},$$

where  $\{w_{ij}\} \triangleq \{\nabla_{ij} u - A_{ij}\}$  denotes the augmented Hessian matrix, and  $\{w^{ij}\}$  denotes the inverse of the matrix  $\{w_{ij}\}$ .

Now we consider the following linear operators of  $F$

$$(2.9) \quad L = w^{ij}(\nabla_{ij} - \nabla_{p_k} A_{ij}(\cdot, u, \nabla u) \nabla_k),$$

and

$$(2.10) \quad \mathcal{L} = L - \nabla_{p_k} \tilde{B} \nabla_k.$$

For convenience, we denote  $\nabla_{\xi\eta} u \triangleq \nabla_{ij} u \xi_i \eta_j$ ,  $w_{\xi\eta} \triangleq w_{ij} \xi_i \eta_j = \nabla_{ij} u \xi_i \eta_j - A_{ij} \xi_i \eta_j$  for any vectors  $\xi$  and  $\eta$ . As usual,  $C$  denotes a constant depending on the known data and may change from line to line in the context.

We begin with a comparison principle of the Neumann boundary value problem for the Monge-Ampère type equation. We set

$$(2.11) \quad \begin{aligned} \mathfrak{F}[u] &= \det Mu - B(x, u, \nabla u), & \text{for } x \in \Omega, \\ G(u) &= \nabla_\nu u - \varphi(x, u), & \text{for } x \in \partial\Omega. \end{aligned}$$

Recall that  $Mu = [\nabla^2 u - A(x, u, \nabla u)]g^{-1}$  and a function  $u$  is called an elliptic function of (1.1) if  $Mu > 0$ . We recall the following comparison principle.

**Lemma 2.1.** *Let  $u, v$  be two elliptic functions of equation (1.1) satisfying*

$$(2.12) \quad \mathfrak{F}[u] \geq \mathfrak{F}[v] \quad x \in \Omega,$$

$$(2.13) \quad G(u) \geq G(v) \quad x \in \partial\Omega.$$

*Assume that  $A$  or  $B$  are strictly increasing in  $z$  and  $G$  is strictly decreasing in  $z$ . Then we have*

$$(2.14) \quad u \leq v, \text{ for } x \in \bar{\Omega}.$$

*Proof.* Set  $w = u - v$ . By a direct calculation, from (2.12), we have

$$(2.15) \quad \begin{aligned} 0 &\leq \mathfrak{F}[u] - \mathfrak{F}[v] \\ &= (\det Mu - \det Mv) - [B(x, u, \nabla u) - B(x, v, \nabla v)] \\ &= \int_0^1 \frac{d}{dt} \det[Mv + t(Mu - Mv)] dt - [B(x, u, \nabla u) - B(x, v, \nabla v)] \\ &= a^{ij} [\nabla_{ij}(u - v) - \nabla_{p_k} A_{ij} \nabla_k(u - v) - \nabla_z A_{ij}(u - v)] \\ &\quad - \nabla_{p_k} B \nabla_k(u - v) - \nabla_z B(u - v) \\ &= a^{ij} \nabla_{ij} w + b^k \nabla_k w + cw, \end{aligned}$$

where  $a^{ij} = \int_0^1 C_t^{ij} dt$  and  $C_t^{ij}$  is the cofactor of the element  $[Mv + t(Mu - Mv)]$ ,  $b^k = -(a^{ij} \nabla_{p_k} A_{ij} + \nabla_{p_k} B)$ ,  $c = -(a^{ij} \nabla_z A_{ij} + \nabla_z B)$ . From the boundary condition (2.13), we have

$$(2.16) \quad \begin{aligned} 0 &\leq G(u) - G(v) \\ &= \nabla_\nu(u - v) - \varphi(x, u) + \varphi(x, v) \\ &= \nabla_\nu(u - v) - \varphi_z(x, \hat{u})(u - v) \\ &= \nabla_\nu w - \varphi_z w, \end{aligned}$$

where  $\hat{u} = \lambda u + (1 - \lambda)v$  for some  $\lambda \in (0, 1)$  appearing by the mean value theorem. Note that the operator  $\tilde{L} = a_{ij} D_{ij} + b^k D_k + c$  is linear and uniformly elliptic. Furthermore, by the monotonicity of both  $A$  and  $B$ , we have  $c \leq 0$ . Since  $\varphi$  is strictly increasing, we have  $\varphi_z > 0$  on  $\partial\Omega$ . Then by Lemma 1.2 in [15],  $w \leq 0$  in  $\bar{\Omega}$ , which leads to the conclusion (2.14). □

From the comparison principle for the Neumann problem (1.1)-(1.2), we infer the uniqueness of the solution of the problem (1.1)-(1.2) immediately.

Since we assume the existence of a  $C^2$  supersolution  $\bar{u}$  satisfying (1.10)-(1.11) and a  $C^2$  subsolution  $\underline{u}$  satisfying (1.12)-(1.13), on the basis of Lemma 2.1, we already have an upper bound for the solution  $u$ , that is  $u \leq \bar{u}$  and a lower bound for the solution  $u$ , that is  $u \geq \underline{u}$ .

Next, we establish the gradient bound for elliptic solution in  $\Omega$  satisfying the Neumann boundary condition. We omit its proof since it has been finished in our previous paper [8].

**Theorem 2.1.** *Let  $\Omega$  be a compact domain in  $(M^n, g)$  with smooth boundary, and  $u$  be a degenerate elliptic solution of the Neumann problem (1.1)-(1.2). Assume  $A$  satisfies the structure condition*

$$(2.17) \quad A(x, u, \nabla u) \geq -\mu_0(1 + |\nabla u|^2)g,$$

*for all  $x \in \bar{\Omega}$  and some positive constant  $\mu_0$ . Then we have the gradient estimate*

$$(2.18) \quad \sup_{\bar{\Omega}} |\nabla u| \leq C,$$

*where  $C$  depends on  $n, g, \mu_0, \Omega, \varphi$  and  $\sup_{\bar{\Omega}} |u|$ .*

### 3. SECOND DERIVATIVE ESTIMATES

In this section, we shall employ a delicate auxiliary function for our discussion to derive the second order derivative estimate and complete the proof of Theorem 1.1. Note that we only need to get an upper bound for the second derivative, since the lower bound can be derived from the ellipticity condition  $\nabla^2 u - A > 0$ . The interior bound can be similarly derived as the interior Pogorelev estimate in [5, 20]. While in the neighbourhood of the boundary, the proof is specific for the Neumann boundary value problem as in [19]. Through out this section, we take full advantage of the assumed supersolution  $\bar{u}$ .

For the arguments below, we assume the functions  $\varphi, \nu$  can be smoothly extended to  $\bar{\Omega} \times \mathbb{R}$  and  $\bar{\Omega}$  respectively. We also assume that near the boundary,  $\nu$  is extended to be constant in the normal directions.

Before we deal with the second derivative estimate, we recall a fundamental lemma in [10, 12], which is also crucial to construct the second derivative estimate.

**Lemma 3.1.** *Suppose that  $u$  is an elliptic solution of (1.1), and  $\bar{u}$  is a strict elliptic supersolution of (1.1). If  $A$  is regular, then*

$$(3.1) \quad \mathcal{L}(e^{K(\bar{u}-u)}) \geq \varepsilon \sum_i w^{ii} - C$$

*holds in  $B_r(x_0)$  for some positive constant  $K$  and uniform positive constant  $\varepsilon$ , where  $C$  is a positive constance depending on  $n, g, A, B, \Omega, \bar{u}$  and  $|u|_{1;\Omega}$ .*

*Proof.* Since  $\bar{u}$  is a strict elliptic supersolution, then there exists  $\varepsilon > 0$  such that  $\bar{u}_\varepsilon = \bar{u} - \varepsilon d^2$  is still a supersolution of (1.1), i.e.

$$(3.2) \quad F(\bar{u}_\varepsilon) \leq \tilde{B}(x, \bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon).$$

Let  $v_\varepsilon = \bar{u}_\varepsilon - u$ , then we have at  $x_0$

$$(3.3) \quad L(\bar{u} - u) = Lv_\varepsilon + \varepsilon Ld^2 \geq Lv_\varepsilon + \varepsilon \sum_i w^{ii}.$$

By the definition of  $L$ , we have

$$(3.4) \quad Lv_\varepsilon = w^{ij}(\nabla_{ij}v_\varepsilon - \nabla_{p_k}A_{ij}(x, u, \nabla u)\nabla_k v_\varepsilon).$$

By the concavity of  $F$ , we have

$$(3.5) \quad F(\bar{u}_\varepsilon) - F(u) \leq w^{ij}[\nabla_{ij}v_\varepsilon - A_{ij}(x, \bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon) + A_{ij}(x, u, \nabla u)],$$

Combining (3.4) and (3.5), we get

$$(3.6) \quad \begin{aligned} Lv_\varepsilon &\geq F(\bar{u}_\varepsilon) - F(u) + w^{ij}[A_{ij}(x, \bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon) - A_{ij}(x, u, \nabla u) \\ &\quad - \nabla_{p_k} A_{ij}(x, u, \nabla u) \nabla_k v_\varepsilon] \\ &= F(\bar{u}_\varepsilon) - F(u) + w^{ij}[A_{ij}(x, \bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon) - A_{ij}(x, u, \nabla \bar{u}_\varepsilon) \\ &\quad + A_{ij}(x, u, \nabla \bar{u}_\varepsilon) - A_{ij}(x, u, \nabla u) - \nabla_{p_k} A_{ij}(x, u, \nabla u) \nabla_k v_\varepsilon]. \end{aligned}$$

From (1.5),

$$(3.7) \quad \begin{aligned} &w^{ij}[A_{ij}(x, \bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon) - A_{ij}(x, u, \nabla \bar{u}_\varepsilon)] \\ &= w^{ij} \nabla_z A_{ij}(x, \hat{z}, \nabla \bar{u}_\varepsilon) v_\varepsilon \geq 0, \end{aligned}$$

where  $u \leq \hat{z} \leq \bar{u}_\varepsilon$ . By the Taylor expansion, we have

$$(3.8) \quad \begin{aligned} &w^{ij}[A_{ij}(x, u, \nabla \bar{u}_\varepsilon) - A_{ij}(x, u, \nabla u) - \nabla_{p_k} A_{ij}(x, u, \nabla u) \nabla_k v_\varepsilon] \\ &= \frac{1}{2} w^{ij} \nabla_{p_k p_l} A_{ij}(x, u, p_\theta) \nabla_k v_\varepsilon \nabla_l v_\varepsilon, \end{aligned}$$

here  $p_\theta = \theta \nabla \bar{u}_\varepsilon + (1 - \theta) \nabla u$  with  $0 \leq \theta \leq 1$ . Let  $v = \bar{u} - u$ . Combining (3.3)-(3.8), we have

$$(3.9) \quad Lv \geq \varepsilon \sum_i w^{ii} + \frac{1}{2} w^{ij} \nabla_{p_k p_l} A_{ij}(x, u, p_\theta) \nabla_k v \nabla_l v - C_1$$

at  $x_0$ , where  $C_1$  is positive constance depend on  $B$ ,  $|u|_{C^1}$  and  $|\bar{u}|_{C^2}$ . By a direct calculation, we have

$$(3.10) \quad \begin{aligned} Le^{Kv} &= Ke^{Kv} [Lv + Kw^{ij} \nabla_i v \nabla_j v] \\ &\geq Ke^{Kv} [\varepsilon \sum_i w^{ii} + \frac{1}{2} w^{ij} \nabla_{p_k p_l} A_{ij}(x, u, p_\theta) \nabla_k v \nabla_l v + Kw^{ij} \nabla_i v \nabla_j v - C_1]. \end{aligned}$$

We assume  $e_1 = \frac{\nabla v}{|\nabla v|}$  when  $\nabla v \neq 0$  at  $x_0$ , or else we finish the proof. Since  $A$  is regular by (1.3), it follows

$$(3.11) \quad \begin{aligned} &\frac{1}{2} w^{ij} \nabla_{p_k p_l} A_{ij} \nabla_k v \nabla_l v + Kw^{ij} \nabla_i v \nabla_j v \\ &= (\frac{1}{2} w^{ij} \nabla_{p_1 p_1} A_{ij} + Kw^{11}) |\nabla v|^2 \\ &\geq (\frac{1}{2} w^{11} \nabla_{p_1 p_1} A_{11} + \sum_{i \neq 1} w^{1j} \nabla_{p_1 p_1} A_{1j} + Kw^{11}) |\nabla v|^2. \end{aligned}$$

Since

$$(3.12) \quad |w^{1j}| \leq w^{11} w^{jj},$$

by the cauchy inequality, we have

$$(3.13) \quad |w^{1j}| \leq \epsilon_0 w^{ii} + \frac{1}{\epsilon_0} w^{11},$$

for positive constant  $\epsilon_0$ . Hence,

$$(3.14) \quad \begin{aligned} Le^{Kv} &\geq Ke^{Kv}[\epsilon \sum_i w^{ii} - \frac{1}{2}\epsilon_0 w^{ii} |\nabla_{p_1 p_1} A_{1i}| |\nabla_1 v|^2 \\ &\quad - \frac{1}{8\epsilon_0} w^{11} |\nabla_{p_1 p_1} A_{1i}| |\nabla v|^2 + Kw^{11} |\nabla_1 v|^2 - C_1]. \end{aligned}$$

Furthermore choosing  $\epsilon_0 \leq \frac{\epsilon}{|\nabla_{p_1 p_1} A_{1i}| |\nabla_1 v|^2}$  and  $K \geq \frac{|\nabla_{p_1 p_1} A_{1i}|}{8\epsilon_0}$ , we have

$$(3.15) \quad \begin{aligned} \mathcal{L}e^{Kv} &= Le^{Kv} - \nabla_{p_k} \tilde{B} \nabla_k e^{Kv} \\ &\geq Ke^{Kv} \frac{\epsilon}{2} \sum_i w^{ii} - C_2 \\ &\geq \epsilon_1 \sum_i w^{ii} - C_3, \end{aligned}$$

where  $\epsilon_1 = \frac{\epsilon}{2} Ke^{Kv}$ . □

Define  $\Omega_\mu = \{x \in \Omega \mid r(x) := \text{dist}(x, \partial\Omega) < \mu\}$ , where  $\mu$  is a positive constant. Here we also assume  $\mu$  is small enough such that  $d(x)$  is smooth in  $\Omega_\mu$ . We assume that the unit inner normal vector  $\nu$  has been smoothly extended from  $\partial\Omega$  to  $\overline{\Omega_\mu}$ , which can be simply achieved by taking  $\nu = \nabla r$  in  $\Omega_\mu$ .

**Lemma 3.2.** *Suppose that  $u$  is an elliptic solution of (1.1), and  $\Omega$  is uniformly  $A$ -convex (1.8). Then*

$$(3.16) \quad \nabla_{\nu\nu}^2 u \leq C(1 + M_2)^{\frac{n-2}{n-1}}$$

on  $\partial\Omega$ , where  $M_2 = \sup_\Omega |\nabla^2 u|$ , and  $C$  is a positive constance depending on  $n, g, A, B, \Omega, \varphi, \delta_0$  and  $|u|_{1;\Omega}$ .

*Proof.* Fixing  $x_0 \in \partial\Omega$ , let  $\{e_1, e_2, \dots, e_n\}$  be a local orthonormal frame near  $x_0$ . By a direct calculation, one has

$$(3.17) \quad \begin{aligned} L(\nabla_\nu u) &= w^{ij} [\nabla_{ijk} u \nu_k + 2\nabla_{ki} u \nabla_j \nu_k + \nabla_k u \nabla_{ij} \nu_k \\ &\quad - \nabla_{pl} A_{ij} \nabla_{kl} u \nu_k - \nabla_{pl} A_{ij} \nabla_k u \nabla_l \nu_k]. \end{aligned}$$

Differentiating equation (1.1) along  $\nu$ , we get

$$(3.18) \quad w^{ij} \nabla_\nu w_{ij} = \nabla_\nu \tilde{B} + \nabla_z \tilde{B} \nabla_\nu u + \nabla_{p_k} \tilde{B} \nabla_{ik} u \nu_i.$$

By the Ricci identity (2.5), it follows

$$(3.19) \quad \begin{aligned} \nabla_\nu w_{ij} &= \nabla_{kij} u \nu_k - \nabla_\nu A_{ij} - \nabla_z A_{ij} \nabla_\nu u - \nabla_{p_k} A_{ij} \nabla_{ik} u \nu_i \\ &= \nabla_{ijk} u \nu_k + R_{lij} \nabla_l u \nu_k - \nabla_\nu A_{ij} - \nabla_z A_{ij} \nabla_\nu u - \nabla_{p_k} A_{ij} \nabla_{ik} u \nu_i. \end{aligned}$$

Since  $w_{ij} = \nabla_{ij} u - A_{ij}$ , we can obtain

$$(3.20) \quad w^{ij} \nabla_{ki} u = w^{ij} (w_{ki} + A_{ki}) = \delta_{jk} + w^{ij} A_{ki}.$$

Putting (3.18), (3.19) and (3.20) into (3.17), we have

$$(3.21) \quad L(\nabla_\nu u) \leq C(1 + \sum_i w^{ii} + |\nabla^2 u|),$$

here  $C$  depend on  $n, g, A, B, \Omega$ , and  $|u|_{1;\Omega}$ . Consider  $h = \nabla_\nu u - \varphi(x, u)$ . From a similar computation, we get

$$(3.22) \quad Lh \leq C(1 + \sum_i w^{ii} + |\nabla^2 u|).$$



From the positivity of  $B$ , we have

$$(3.23) \quad 1 \leq Cw^{ii}, \quad (w_{ii})^{\frac{1}{n-1}} \leq C(w^{ii}).$$

So

$$(3.24) \quad Lh \leq C(1 + M_2^{\frac{n-2}{n-1}}) \sum_i w^{ii}.$$

Since the domain  $\Omega$  is  $A$ -convexity (1.8),

$$(3.25) \quad L\phi \geq \delta_0 \sum_i w^{ii}.$$

Choosing  $-\phi$  as a barrier function, a standard barrier argument leads to

$$(3.26) \quad \nabla_\nu h \leq C(1 + M_2^{\frac{n-2}{n-1}}),$$

which completes the proof of the lemma.  $\square$

Applying the tangential operator to the boundary condition (1.2), we have the mixed tangential normal derivative estimate. Then from Lemma 3.2, the double normal derivative estimate has been bounded. Next, we shall adopt the method in [19] to obtain the double tangential derivative bound on the boundary. Consequently we achieve the second derivative estimate on the boundary.

Modifying the elliptic supersolution  $\bar{u}$  by adding a perturbation function  $-a\phi$ , where  $a$  is a small positive constant. Note that if  $a$  is small enough then the function  $\bar{u} - a\phi$  is still elliptic supersolution of (1.1) and (1.2). On  $\partial\Omega$ , we have

$$(3.27) \quad \nabla_\nu(\bar{u} - a\phi - u) \geq a,$$

by the condition (1.7).

We now consider an auxiliary function  $V(x, \xi)$  given by

$$(3.28) \quad V(x, \xi) = e^{\alpha|\nabla(u-\lambda\phi)|^2 + \beta\Phi}(w_{\xi\xi} - V'(x, \xi))$$

for  $x \in \bar{\Omega}$ ,  $\xi \in T_x M$  with  $|\xi| = 1$ ,

$$\lambda = \max_{\bar{\Omega}} |\nabla u|,$$

and

$$(3.29) \quad V'(x, \xi) = 2g(\xi, \nu)[\nabla_{\xi'}\varphi(x, u) - g(\nabla u, \xi') - A_{\nu\xi'}],$$

where  $\xi' = \xi - g(\xi, \nu)\nu$ ,  $\nu$  denote the extension of the inner normal vector field on  $M$  and  $\Phi = e^{K(\bar{u}-u-a\phi)}$ . We assume that  $V$  attain its maximum at  $(x_0, \xi)$ .

**Case 1.**  $x_0$  is an interior point.  $\xi$  still denotes the extension of  $\xi$  in a small neighborhood of  $x_0$  with  $\nabla\xi(x_0) = 0$ , and let  $\{e_1, e_2, \dots, e_n\}$  be a local orthonormal frame in the neighborhood with  $w_{ij}$  diagonal at  $x_0$  and  $w_{11}$  is the largest eigenvalue. Set  $H = \ln V$ , then we have at  $x_0$ ,

$$(3.30) \quad 0 = \nabla_i H = \frac{\nabla_i(w_{\xi\xi} - V')}{w_{\xi\xi} - V'} + 2\alpha\nabla_k(u - \lambda\phi)\nabla_{ik}(u - \lambda\phi) + \beta\nabla_i\Phi, \quad \text{for } i = 1 \dots n,$$

$$(3.31) \quad 0 \geq \mathcal{L}H = \mathcal{L}\ln(w_{\xi\xi} - V') + 2\alpha\mathcal{L}|\nabla(u - \lambda\phi)|^2 + \beta\mathcal{L}\Phi.$$

By a direct calculation, we have

$$\begin{aligned}
(3.32) \quad \mathcal{L} \ln(w_{\xi\xi} - V') &= \frac{\mathcal{L}(w_{\xi\xi} - V')}{w_{\xi\xi} - V'} - \frac{w^{ij} \nabla_i (w_{\xi\xi} - V') \nabla_j (w_{\xi\xi} - V')}{(w_{\xi\xi} - V')^2} \\
&\geq \frac{\mathcal{L} w_{\xi\xi} - \mathcal{L} V'}{w_{\xi\xi} - V'} - (1 + \theta) \frac{w^{ij} \nabla_i w_{\xi\xi} \nabla_j w_{\xi\xi}}{(w_{\xi\xi} - V')^2} \\
&\quad - C(\theta) \frac{w^{ij} \nabla_i V' \nabla_j V'}{(w_{\xi\xi} - V')^2}.
\end{aligned}$$

From the definition of  $\mathcal{L}$ , we have

$$(3.33) \quad \mathcal{L} w_{\xi\xi} = w^{ij} [\nabla_{ij} w_{\xi\xi} - \nabla_{p_k} A_{ij} \nabla_k w_{\xi\xi}] - \nabla_{p_k} \tilde{B} \nabla_k w_{\xi\xi}.$$

Taking derivative on both sides of the equation (2.8) in the direction of  $\xi$ , we get

$$(3.34) \quad w^{ij} \nabla_\xi w_{ij} = \nabla_\xi \tilde{B} + \nabla_z \tilde{B} \nabla_\xi u + \nabla_{p_k} \tilde{B} \nabla_{jk} u \xi_j.$$

A further differentiation in the direction of  $\xi$  yields

$$\begin{aligned}
(3.35) \quad w^{ij} \nabla_\xi w_{ij} &= w^{ik} w^{jl} \nabla_\xi w_{ij} \nabla_\xi w_{kl} + \nabla_{p_k} \tilde{B} \nabla_{ijk} u \xi_i \xi_j + \nabla_z \tilde{B} \nabla_{ij} u \xi_i \xi_j \\
&\quad + \nabla_{\xi\xi} \tilde{B} + 2 \nabla_{\xi z} \tilde{B} \nabla_\xi u + 2 \nabla_{\xi p_k} \tilde{B} \nabla_{jk} u \xi_j + \nabla_{zz} \tilde{B} (\nabla_\xi u)^2 \\
&\quad + 2 \nabla_{zp_k} \tilde{B} \nabla_\xi u \nabla_{jk} u \xi_j + \nabla_{p_k p_l} \tilde{B} \nabla_{jk} u \nabla_{il} u \xi_i \xi_j
\end{aligned}$$

at  $x_0$ . Then

$$(3.36) \quad w^{ij} \nabla_\xi w_{ij} \geq w^{ik} w^{jl} \nabla_\xi w_{ij} \nabla_\xi w_{kl} + \nabla_{p_k} \tilde{B} \nabla_{ijk} u \xi_i \xi_j - C[1 + (w_{ii})^2].$$

For convenient, we define a (0,3)-tensor as follows,

$$\begin{aligned}
(3.37) \quad T_{ijk} &= \nabla_k A_{ij} + \nabla_z A_{ij} \nabla_k u + \nabla_{p_l} A_{ij} \nabla_{kl} u \\
&\quad - \nabla_j A_{ik} - \nabla_z A_{ik} \nabla_j u - \nabla_{p_l} A_{ik} \nabla_{jl} u.
\end{aligned}$$

Besides, by the Ricci identity, we have

$$(3.38) \quad \nabla_k w_{ij} - \nabla_j w_{ik} = \nabla_s u R_{sijk} - T_{ijk}.$$

By a direct computation, it follows

$$\begin{aligned}
(3.39) \quad &w^{ij} [\nabla_{ij} w_{\xi\xi} - \nabla_{\xi\xi} w_{ij}] \\
&= w^{ij} [\nabla_{ij} w_{kl} - \nabla_{kl} w_{ij}] \xi_k \xi_l + 2 w^{ij} w_{kl} \nabla_{ij} \xi_k \xi_l,
\end{aligned}$$

and from the Ricci identity and (3.38),

$$\begin{aligned}
(3.40) \quad \nabla_{ij} w_{kl} &= \nabla_{jl} w_{ki} - \nabla_j T_{kli} + \nabla_{js} u R_{skli} + \nabla_s u \nabla_j R_{skli} \\
&= \nabla_{lj} w_{ik} + w_{si} R_{sklj} + w_{sk} R_{silj} - \nabla_j T_{kli} \\
&\quad + \nabla_{js} u R_{skli} + \nabla_s u \nabla_j R_{skli} \\
&= \nabla_{lk} w_{ij} - \nabla_l T_{ikj} - \nabla_j T_{kli} + \nabla_{ls} u R_{sikj} \\
&\quad + \nabla_s u \nabla_l R_{sikj} + \nabla_{js} u R_{skli} + \nabla_s u \nabla_j R_{skli} \\
&\quad + w_{si} R_{sklj} + w_{sk} R_{silj}.
\end{aligned}$$

Combining (3.36), (3.39) and (3.40), we have

$$\begin{aligned}
(3.41) \quad &w^{ij} \nabla_{ij} w_{\xi\xi} - \nabla_{p_k} \tilde{B} \nabla_k w_{\xi\xi} \\
&\geq w^{ik} w^{jl} \nabla_\xi w_{ij} \nabla_\xi w_{kl} - w^{ij} (\nabla_l T_{ikj} + \nabla_j T_{kli}) \xi_k \xi_l - C[1 + \mathcal{T} w_{ii} + (w_{ii})^2].
\end{aligned}$$

where  $\mathcal{T} = w^{ii}$ . From the definition of the tensor  $T$  given by (3.37), it follows

$$\begin{aligned}
& w^{ij}(\nabla_l T_{ikj} + \nabla_j T_{kli})\xi_k \xi_l \\
& = w^{ij}(\nabla_{p_s} A_{kl} \nabla_{jis} u - \nabla_{p_s} A_{ij} \nabla_{lks} u)\xi_k \xi_l + w^{ij}(A_{sk} R_{sijl} + A_{si} R_{skjl})\xi_k \xi_l \\
& + w^{ij}\{(\nabla_z A_{kl} \nabla_{ij} u + \nabla_{ij} A_{kl} + 2\nabla_{iz} A_{kl} \nabla_j u + 2\nabla_{ip_s} A_{kl} \nabla_{js} u \\
& + \nabla_{zz} A_{kl} \nabla_i u \nabla_j u + 2\nabla_{p_s z} A_{kl} \nabla_{sj} u \nabla_i u + \nabla_{p_s p_t} A_{kl} \nabla_{is} u \nabla_{jt} u) \\
& - (\nabla_z A_{ij} \nabla_{kl} u + \nabla_{kl} A_{ij} + 2\nabla_{kz} A_{ij} \nabla_l u + 2\nabla_{kp_s} A_{ij} \nabla_{ls} u \\
& + \nabla_{zz} A_{ij} \nabla_k u \nabla_l u + 2\nabla_{p_s z} A_{ij} \nabla_{sl} u \nabla_k u + \nabla_{p_s p_t} A_{ij} \nabla_{ks} u \nabla_{lt} u)\}\xi_k \xi_l.
\end{aligned} \tag{3.42}$$

By a direct calculation, we have

$$w^{ij} \nabla_{p_s} A_{ij} \nabla_{lks} u \xi_k \xi_l = w^{ij} \nabla_{p_s} A_{ij} (\nabla_s w_{\xi\xi} + \nabla_s A_{\xi\xi} + \nabla_m u R_{mksl}), \tag{3.43}$$

and

$$\begin{aligned}
& w^{ij} \nabla_{jis} u = w^{ij} \nabla_{sij} u + w^{ij} R_{misj} \nabla_m u \\
& = w^{ij} \nabla_s w_{ij} + w^{ij} (\nabla_s A_{ij} + \nabla_z A_{ij} \nabla_s u + \nabla_{p_m} A_{ij} \nabla_{ms} u + R_{misj} \nabla_m u) \\
& = \nabla_s \tilde{B} + \nabla_z \tilde{B} \nabla_s u + \nabla_{p_m} \tilde{B} \nabla_{sm} u + w^{ij} (\nabla_s A_{ij} + \nabla_z A_{ij} \nabla_s u \\
& + \nabla_{p_m} A_{ij} \nabla_{ms} u + R_{misj} \nabla_m u).
\end{aligned} \tag{3.44}$$

So from (3.42), (3.43) and (3.44), we get

$$\begin{aligned}
& w^{ij}(\nabla_l T_{ikj} + \nabla_j T_{kli})\xi_k \xi_l \\
& \leq -w^{ij} \nabla_{p_s} A_{ij} \nabla_s w_{\xi\xi} + C(\mathcal{T} + \mathcal{T} w_{ii} + 1).
\end{aligned} \tag{3.45}$$

Then we have by (3.41) and (3.45)

$$\mathcal{L} w_{\xi\xi} \geq w^{ik} w^{jl} \nabla_\xi w_{ij} \nabla_\xi w_{kl} - C[(1 + w_{ii})\mathcal{T} + (w_{ii})^2]. \tag{3.46}$$

From a similar argument, we can also have

$$|\mathcal{L} V'| \leq C[(1 + w_{ii})\mathcal{T} + (w_{ii})^2], \tag{3.47}$$

and

$$\begin{aligned}
& \frac{1}{2} \mathcal{L} |\nabla(u - \lambda\phi)|^2 = w^{ij} [\nabla_{ijk}(u - \lambda\phi) \nabla_k(u - \lambda\phi) + \nabla_{ik}(u - \lambda\phi) \nabla_{jk}(u - \lambda\phi) \\
& - \nabla_{p_s} A_{ij} \nabla_{sk}(u - \lambda\phi) \nabla_k(u - \lambda\phi)] - \nabla_{p_s} \tilde{B} \nabla_{sk}(u - \lambda\phi) \nabla_s(u - \lambda\phi).
\end{aligned} \tag{3.48}$$

By a direct calculation, it follows

$$\begin{aligned}
& w^{ij} \nabla_{ik}(u - \lambda\phi) \nabla_{jk}(u - \lambda\phi) \\
& = w^{ij} (w_{ik} + A_{ik} - \lambda \nabla_{ik} \phi) (w_{jk} + A_{jk} - \lambda \nabla_{jk} \phi) \\
& = w_{ii} + 2A_{ii} - 2\lambda \Delta \phi + w^{ij} (A_{ik} - \lambda \nabla_{ik} \phi) (A_{jk} - \lambda \nabla_{jk} \phi),
\end{aligned} \tag{3.49}$$

and

$$\begin{aligned}
& w^{ij} \nabla_{ijk}(u - \lambda\phi) \nabla_k(u - \lambda\phi) \\
& = w^{ij} \nabla_{ijk} u \nabla_k(u - \lambda\phi) - \lambda w^{ij} \nabla_{ijk} \phi \nabla_k(u - \lambda\phi) \\
& = w^{ij} (\nabla_k w_{ij} + R_{sikj} \nabla_s u + \nabla_k A_{ij} + \nabla_z A_{ij} \nabla_k u + \nabla_{p_s} A_{ij} \nabla_{ks} u) \nabla_k(u - \lambda\phi) \\
& - \lambda w^{ij} \nabla_{ijk} \phi \nabla_k(u - \lambda\phi).
\end{aligned} \tag{3.50}$$

Putting (3.49), (3.50) and (3.34) into (3.48), we have

$$\frac{1}{2} \mathcal{L} |\nabla(u - \lambda\phi)|^2 \geq w_{ii} - C\mathcal{T}. \tag{3.51}$$

Combining (3.31), (3.32), (3.46) and (3.51), we get from Lemma 3.1,

$$(3.52) \quad \begin{aligned} 0 \geq & \frac{w^{ik}w^{jl}\nabla_\xi w_{ij}\nabla_\xi w_{kl}}{w_{\xi\xi} - V'} - (1 + \theta) \frac{w^{ij}\nabla_i w_{\xi\xi}\nabla_j w_{\xi\xi}}{(w_{\xi\xi} - V')^2} \\ & - \frac{C[(1 + w_{ii})\mathcal{T} + (w_{ii})^2]}{w_{\xi\xi} - V'} - C(\theta) \frac{w^{ij}\nabla_i V'\nabla_j V'}{(w_{\xi\xi} - V')^2} \\ & + 2\alpha w_{ii} + (\beta - 2\alpha C)\mathcal{T} - \beta C. \end{aligned}$$

Since  $w_{11}$  is the largest eigenvalue, then from the inequality in [19], we get at  $x_0$ ,

$$(3.53) \quad w^{ik}w^{jl}\nabla_\xi w_{ij}\nabla_\xi w_{kl} \geq \frac{1}{w_{11}}w^{jl}\nabla_\xi w_{li}\nabla_\xi w_{jk}\xi_k\xi_i.$$

From (3.38), we have

$$(3.54) \quad \nabla_\xi w_{jk}\xi_k = \nabla_j w_{\xi\xi} + (\nabla_s u R_{skji} - T_{kji})\xi_k\xi_i.$$

Then

$$(3.55) \quad w^{ik}w^{jl}\nabla_\xi w_{ij}\nabla_\xi w_{kl} \geq \frac{1 - \theta}{w_{11}}w^{jl}\nabla_j w_{\xi\xi}\nabla_l w_{\xi\xi} - \frac{C_\theta}{w_{11}}(1 + w_{ii}).$$

Since  $V'$  is bounded, one can define a quantity as follows

$$M_1 = \sup\{V'(x_0, \eta) | \eta \in T_{x_0}M, |\eta| = 1\}.$$

For

$$w_{11} \geq w_{\xi\xi}, \quad \text{and} \quad w_{\xi\xi} - V'(x_0, \xi) \geq w_{11} - V'(x_0, e_1),$$

then if

$$(3.56) \quad w_{11} > \frac{M_1}{\theta},$$

we have

$$(3.57) \quad |w_{11} - w_{\xi\xi} + V'(x_0, \xi)| < \theta w_{11}.$$

We assume (3.56) holds, or else we get the bound for  $w$ , and then

$$(3.58) \quad \begin{aligned} & \frac{w^{ik}w^{jl}\nabla_\xi w_{ij}\nabla_\xi w_{kl}}{w_{\xi\xi} - V'} - (1 + \theta) \frac{w^{ij}\nabla_i w_{\xi\xi}\nabla_j w_{\xi\xi}}{(w_{\xi\xi} - V')^2} \\ & \geq - \frac{3\theta}{(1 - \theta)^2} \frac{w^{ij}\nabla_i w_{\xi\xi}\nabla_j w_{\xi\xi}}{w_{11}^2} - \frac{C_\theta}{w_{11}^2}(1 + w_{ii}). \end{aligned}$$

By the definition of  $V'$ , we have

$$(3.59) \quad |\nabla V'| \leq C(1 + w_{ii}).$$

Putting (3.58) and (3.59) into (3.52), we get from (3.30) the following

$$(3.60) \quad 0 \geq (2\alpha - C - C\alpha^2\theta)w_{ii} + (\beta - 2\alpha - C - C\beta^2\theta)\mathcal{T} - \beta C.$$

So we obtain the estimate  $w_{ii} \leq C$  by choosing  $\alpha, \beta$  large and fixing a small  $\theta$ .

**Case 2.**  $x_0 \in \partial\Omega$ . In this case, we consider the following three subcases by different directions of  $\xi$ .

*Subcase (i).*  $\xi = \nu$ , we proved in Lemma 3.2 that

$$(3.61) \quad \nabla_{\nu\nu}u \leq C(1 + M_2)^{\frac{n-2}{n-1}}.$$

*Subcase (ii).*  $\xi$  is neither normal nor tangential to  $\partial\Omega$ . The unit vector  $\xi$  can be written as

$$(3.62) \quad \xi = \xi^T + g(\xi \cdot \nu)\nu,$$

here  $\xi^T \in T_{x_0} \partial\Omega$  is the tangential part of  $\xi$ . Let

$$\tau = \frac{\xi^T}{|\xi^T|}.$$

Then by the constructions of  $V$  and  $V'$ , we have

$$(3.63) \quad w_{\xi\xi} = |\xi^T|^2 w(\tau, \tau) + g(\xi \cdot \nu)^2 w(\nu, \nu) + V'(x_0, \xi).$$

So

$$(3.64) \quad \begin{aligned} V(x_0, \xi) &= |\xi^T|^2 V(x_0, \tau) + g(\xi \cdot \nu)^2 V(x_0, \nu) \\ &\leq |\xi^T|^2 V(x_0, \xi) + g(\xi \cdot \nu)^2 V(x_0, \nu), \end{aligned}$$

which implies  $V(x_0, \xi) \leq V(x_0, \nu)$ . In fact,  $V(x_0, \xi) = V(x_0, \nu)$  for  $V(x_0, \xi) \geq V(x_0, \nu)$ .

*Subcase (iii).*  $\xi$  is tangential to  $\partial\Omega$  at  $x_0$ . Let  $\{e_1, e_2, \dots, e_n\}$  be the local orthonormal frame near  $x_0$  on  $\Omega$  by parallel translation of a local orthonormal frame on  $\partial\Omega$  with  $e_n = \nu$ . We still use  $\xi$  denote the extension of  $\xi$  in a small neighborhood of  $x_0$  with  $\nabla \xi(x_0) = 0$ . Then

$$\nabla_n V \leq 0 \quad \text{at } x_0,$$

so by (3.27) we have

$$(3.65) \quad \begin{aligned} 0 &\geq (\alpha \nabla_n |\nabla(u - \lambda\phi)|^2 + \beta \nabla_n \Phi) w_{\xi\xi} - \nabla_n w_{\xi\xi} - \nabla_n V'(x, \xi) \\ &\geq [2\alpha \nabla_{nn}(u - \lambda\phi) \nabla_n(u - \lambda\phi) + 2\alpha \sum_{i=1}^{n-1} \nabla_{in}(u - \lambda\phi) \nabla_i(u - \lambda\phi) + \beta a] w_{\xi\xi} \\ &\quad - \nabla_n w_{\xi\xi} - \nabla_n V'(x, \xi). \end{aligned}$$

From the boundary condition (1.2), it follows

$$(3.66) \quad \begin{aligned} \nabla_{in} u &= \nabla_i \nabla_n u - \nabla_{\nabla_i e_n} u \\ &= \nabla_i \varphi(x, u) \nabla_i u + \nabla_z \varphi(x, u) (\nabla_i u)^2 - \nabla_{\nabla_i e_n} u \nabla_i u, \end{aligned}$$

when  $1 \leq i \leq n-1$ . Since  $w$  is positive definite and  $\lambda \geq |\nabla u|$ , then

$$(3.67) \quad \begin{aligned} \nabla_{nn}(u - \lambda\phi) \nabla_n(u - \lambda\phi) &= (\nabla_n u + \lambda)(w_{nn} + A_{nn} - \lambda \nabla_{nn} \phi) \\ &\geq (\nabla_n u + \lambda)(A_{nn} - \lambda \nabla_{nn} \phi) \\ &\geq -C. \end{aligned}$$

Putting (3.66) and (3.67) into (3.65), we can obtain

$$(3.68) \quad 0 \geq (\beta a - \alpha C) w_{\xi\xi} - \nabla_n w_{\xi\xi} - \nabla_n V'(x, \xi).$$

By a direct calculation, we have

$$(3.69) \quad \begin{aligned} \nabla_\nu w_{\xi\xi} &= (\nabla_{ijk} u + R_{sijk} \nabla_s u) \xi_i \xi_j \nu_k - \nabla_\nu A(\xi, \xi) \\ &= \nabla_{\xi\xi}(\nabla_\nu u) - 2g(\nabla_\xi \nu, \nabla_\xi \nabla u) - g(\nabla_{\xi\xi} \nu, \nabla u) \\ &\quad + R_{sijk} \nabla_s u \xi_i \xi_j \nu_k - \nabla_\nu A(\xi, \xi). \end{aligned}$$

Since  $\xi$  is tangential to  $\partial\Omega$  at  $x_0$ ,

$$(3.70) \quad \nabla_{\xi\xi}(\nabla_\nu u) = \nabla_z \varphi \nabla_{\xi\xi} u + \nabla_{\xi\xi} \varphi + 2\nabla_{\xi z} \varphi \nabla_\xi u + \nabla_{zz} \varphi (\nabla_\xi u)^2,$$

So, we have

$$(3.71) \quad \begin{aligned} \nabla_\nu w_{\xi\xi} &= \nabla_z \varphi \nabla_{\xi\xi} u + \nabla_{\xi\xi} \varphi + \nabla_{zz} \varphi (\nabla_\xi u)^2 + R_{sijk} \nabla_s u \xi_i \xi_j \nu_k \\ &\quad + 2\nabla_{\xi z} \varphi \nabla_\xi u - 2g(\nabla_\xi \nu, \nabla_\xi \nabla u) - g(\nabla_{\xi\xi} \nu, \nabla u) - \nabla_\nu A(\xi, \xi) \\ &\geq -C[1 + w_{\xi\xi}], \end{aligned}$$

and

$$(3.72) \quad |\nabla_\nu V'| \leq C[1 + w_{\xi\xi}].$$

From (3.68), (3.71) and (3.72), we get

$$(3.73) \quad 0 \geq (\beta a - \alpha C - C)w_{\xi\xi} - C.$$

Then we can finish the proof by choosing  $\beta$  large enough.

#### 4. PROOF OF THEOREM 1.2

In this section, we give a brief proof of Theorem 1.2. Since we now have the derivative estimates up to second order, we can use the continuity method to prove our existence theorem.

*Proof of Theorem 1.2.* By the maximum modulus in Section 2 together with Theorem 2.1, we can derive a global second derivative Hölder estimate

$$(4.1) \quad |u|_{2,\alpha;\Omega} \leq C,$$

for elliptic solutions  $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$  of the semilinear Neumann boundary value problem (1.1)-(1.2) for  $0 < \alpha < 1$ . The estimate (4.1) is obtained in [18], Theorem 3.2, (see also [17, 25]). With this  $C^{2,\alpha}$  estimate, we can use the method of continuity, (see [3], Theorem 17.22, Theorem 17.28), to derive the existence of a solution  $u \in C^{2,\alpha}(\bar{\Omega})$ . By virtue of the maximum principles (see [3], Theorem 9.1, Theorem 9.6), the proof of Theorem 1.1 carry over to solution  $u \in W^{4,n}(\Omega) \cap C^3(\bar{\Omega})$ . Thus, from the Schauder theory, (see [3], Section 6.7), we can improve  $C^{2,\alpha}(\bar{\Omega})$  solutions with  $0 < \alpha < 1$  to be in spaces  $W^{4,p}(\Omega) \cap C^{3,\delta}(\bar{\Omega})$  for all  $p < \infty$ ,  $0 < \delta < 1$ . The uniqueness is from the comparison principle in Section 2, see Lemma 2.1.  $\square$

**Acknowledgements.** The authors would like to express their gratitude to the referees for the careful reading of the manuscript and their comments. This work was partially supported by the NSF of China (Grant Nos. 11501184, 11401131 and 11101132).

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